

Gaussian random variable (Gaussian RV)

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Abstract

A standard Gaussian random variable (Gaussian RV) is a real-valued random variable (RV) whose probability density function (pdf) is proportional to $\exp(-\eta^2/2)$. A general Gaussian RV with mean μ and variance σ^2 is obtained by scaling and shifting a standard Gaussian RV; its probability distribution is the normal distribution. A Gaussian random vector arises from applying a matrix to a vector of independent standard Gaussian RVs and adding a mean vector; its probability distribution is the multivariate normal distribution. Gaussian RVs are widely used probabilistic models in the statistical analysis of machine learning (ML) methods, partly due to the central limit theorem (CLT). Among all random vectors with a given covariance matrix, the Gaussian random vector maximizes the differential entropy.

Definition

A standard Gaussian random variable (RV) is a real-valued RV x with a probability density function (pdf) (Bertsekas and Tsitsiklis, 2008; Gray, 2009; Papoulis and Pillai, 2002)

$$p^{(x)}(\eta) = \frac{1}{\sqrt{2\pi}} \exp(-\eta^2/2).$$

Given a standard Gaussian RV x , a general Gaussian RV x' with mean μ and variance σ^2 can be constructed via $x' := \sigma x + \mu$. The probability distribution of a Gaussian RV is referred to as normal distribution, denoted by $\mathcal{N}(\mu, \sigma^2)$.

A Gaussian random vector $\mathbf{x} \in \mathbb{R}^d$ with covariance matrix $\mathbf{C}$ and mean $\boldsymbol{\mu}$ can be constructed as (Gray, 2009; Lapidath, 2017; Papoulis and Pillai, 2002)

$$\mathbf{x} := \mathbf{A}\mathbf{z} + \boldsymbol{\mu},$$

where $\mathbf{z} := (z_1, \dots, z_d)^\top$ is a vector of independent and identically distributed (i.i.d.) standard Gaussian RVs, and $\mathbf{A} \in \mathbb{R}^{d \times d}$ is any matrix satisfying $\mathbf{A}\mathbf{A}^\top = \mathbf{C}$. The probability distribution of a Gaussian random vector is referred to as the multivariate normal distribution, denoted by $\mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$.

A Gaussian random vector $\mathbf{x} = (x_1, \dots, x_d)^\top$ can be interpreted as a stochastic process indexed by the set $\mathcal{I} = \{1, \dots, d\}$. A Gaussian process (GP) is a stochastic process over an arbitrary index set $\mathcal{I}$ such that any restriction to a finite subset $\mathcal{I}' \subseteq \mathcal{I}$ yields a Gaussian random vector (Rasmussen and Williams, 2006).

Gaussian RVs are widely used probabilistic models in the statistical analysis of machine learning (ML) methods. Their significance arises partly from the central limit theorem (CLT), which provides conditions under which the average of many independent RVs (not necessarily Gaussian themselves) tends toward a Gaussian RV (Ross, 2014).

The multivariate normal distribution is also distinct in that it represents maximum uncertainty. Among all random vectors with a given covariance matrix $\mathbf{C}$, the random vector $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$ maximizes differential entropy (Cover and Thomas, 2006, Th. 8.6.5). This makes GPs a natural choice for capturing uncertainty or the lack of (domain) knowledge.

Cross References

multivariate normal distribution, GP, probabilistic model, CLT, differential entropy

References

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