

# convex

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## Abstract

A subset of the Euclidean space is convex if it contains the line segment between any two of its points. A function is convex if its epigraph is a convex set. A convex optimization problem is the minimization of a convex function over a convex set. Empirical risk minimization (ERM) for a convex hypothesis space and a convex loss function is a convex optimization problem. Such an optimization problem can be solved efficiently by iterative methods that use local approximations to update model parameters. The theoretical underpinning of these methods is that any local minimizer of a convex objective function is necessarily also a global minimizer.

## Definition

A subset  $\mathcal{C} \subseteq \mathbb{R}^d$  of the Euclidean space  $\mathbb{R}^d$  is referred to as convex if it contains the line segment between any two points  $\mathbf{w}, \mathbf{w}' \in \mathcal{C}$  in that set, i.e.,

$$\beta \mathbf{w} + (1 - \beta) \mathbf{w}' \in \mathcal{C} \quad \text{for all } \beta \in [0, 1].$$

Similarly, a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is convex if its epigraph  $\{(\mathbf{w}^\top, t)^\top \in \mathbb{R}^{d+1} : t \geq f(\mathbf{w})\}$  is a convex set (Boyd and Vandenberghe, 2004). One example of a convex set and a convex function is illustrated in Fig. 1.

Convexity is a central property for machine learning (ML) methods. For a convex objective function, such as the average squared error loss of linear regression or the objective function of the support vector machine (SVM), every local minimum is a global minimum; for differentiable convex objective functions and suitable step size choices, gradient descent (GD) converges to such a minimum (Boyd and Vandenberghe, 2004).

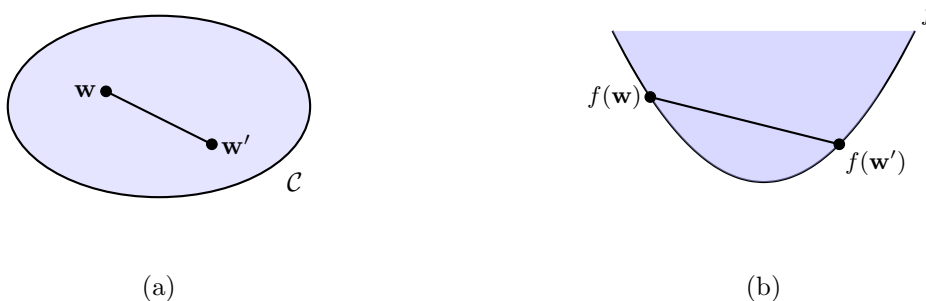


Figure 1: (a) Convex set  $\mathcal{C} \subseteq \mathbb{R}^d$ : the line segment between any two points of  $\mathcal{C}$  stays in  $\mathcal{C}$ . (b) Convex function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ : the shaded epigraph is a convex set, and the line segment connecting any two points on the graph of  $f$  lies above the graph

A closed convex set admits a representation by halfspaces: it is the intersection of all closed halfspaces that contain it (Boyd and Vandenberghe, 2004, Ch. 2). Not all of these halfspaces are needed: those whose boundary hyperplane touches the set suffice, in the following sense. A supporting hyperplane of the set  $\mathcal{C}$  is a hyperplane  $\{\mathbf{w} : \mathbf{a}^\top \mathbf{w} = b\}$ , with normal vector  $\mathbf{a} \neq \mathbf{0}$  and  $b \in \mathbb{R}$ , such that  $\mathbf{a}^\top \mathbf{w} \leq b$  holds for every  $\mathbf{w} \in \mathcal{C}$ , with equality for at least one boundary point of  $\mathcal{C}$  (see supporting hyperplane). The halfspace  $\{\mathbf{w} : \mathbf{a}^\top \mathbf{w} \leq b\}$  then contains  $\mathcal{C}$  and is referred to as a supporting halfspace. A closed convex set is already the intersection of its supporting halfspaces (Rockafellar, 1970). Intersecting only finitely many of them yields an outer approximation of the set (see Fig. 2). For a bounded set and a suitable choice of the halfspaces, e.g., supporting halfspaces at equally spread boundary points, the excess area (volume) of the approximation over the set shrinks toward zero as the number of halfspaces increases. This representation turns geometric statements about convex sets into systems of linear inequalities (Rockafellar, 1970, Sect. 13).

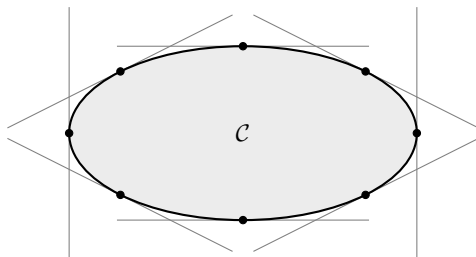


Figure 2: A convex set  $\mathcal{C}$  (shaded) and eight of its supporting hyperplanes (gray lines). Each supporting hyperplane defines a halfspace that contains  $\mathcal{C}$ ; intersecting the eight halfspaces yields an outer approximation of  $\mathcal{C}$ , and the intersection of all such halfspaces recovers  $\mathcal{C}$  exactly

## Cross References

Euclidean space, function, epigraph, convex optimization, minimum, objective function, halfspace, supporting hyperplane

## References

1. Boyd and Vandenberghe (2004). Convex Optimization. Cambridge Univ. Press, Cambridge, U.K.
2. Rockafellar (1970). Convex Analysis. Princeton Univ. Press, Princeton, NJ, USA.